# Applications of Quantified Boolean Formulae Decision Procedures

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# Quantified Boolean Formulae (QBF)



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- propositional logic (SAT $\subseteq$ QBF)
  - constants $0, 1$
  - operators $\wedge, \neg, \rightarrow, \leftrightarrow, \ldots$
  - variables $x, y, \ldots$ over boolean domain $\mathbb{B} = \{0, 1\}$
- quantifiers over boolean variables
  - valid $\forall x[\exists y[x \leftrightarrow y]]$ (read $\leftrightarrow$ as $=$)
  - invalid $\exists x[\forall y[x \leftrightarrow y]]$

- semantics given as expansion of quantifiers

$$\exists x[f] \;\equiv\; f[0/x] \vee f[1/x] \qquad \forall x[f] \;\equiv\; f[0/x] \wedge f[1/x]$$

- expansion as translation from SAT to QBF is exponential
  - SAT problems have only existential quantifiers
  - expansion of universal quantifiers doubles formula size
- most likely no polynomial translation from SAT to QBF
  - otherwise PSPACE = NP

# Tic-Tac-Toe



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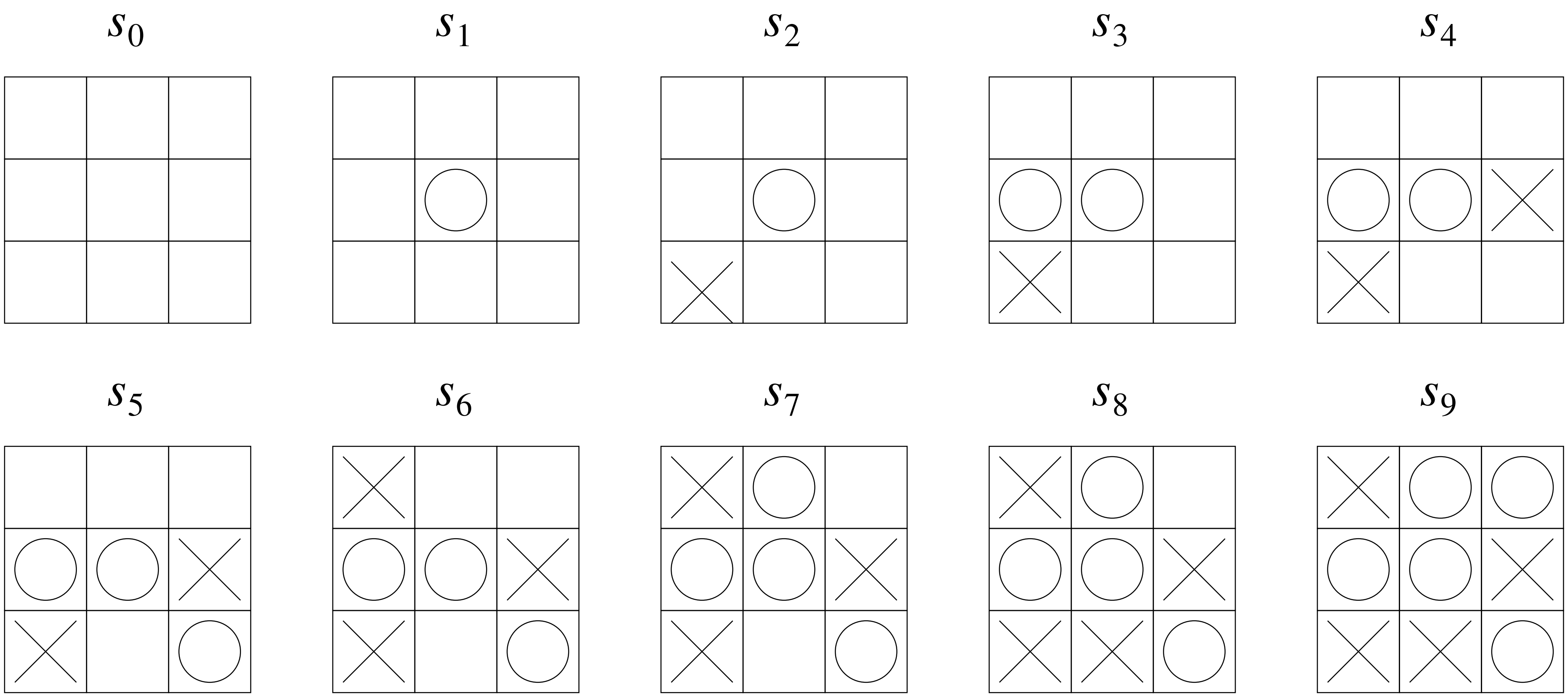

# No Winning Strategy for Tic-Tac-Toe

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$$
\begin{array}{l}
\forall s_0[\mathit{empty}(s_0) \rightarrow \\
\quad \forall x_1[\mathit{circle}(s_0, x_1, s_1) \rightarrow \\
\quad\quad \exists y_2[\mathit{cross}(s_1, y_2, s_2) \wedge \\
\quad\quad\quad \forall x_3[\mathit{circle}(s_2, x_3, s_3) \rightarrow \\
\quad\quad\quad\quad \exists y_4[\mathit{cross}(s_3, y_4, s_4) \wedge \\
\quad\quad\quad\quad\quad \forall x_5[\mathit{circle}(s_4, x_5, s_5) \rightarrow \\
\quad\quad\quad\quad\quad\quad \exists y_6[\mathit{cross}(s_5, y_6, s_6) \wedge \\
\quad\quad\quad\quad\quad\quad\quad \forall x_7[\mathit{circle}(s_6, x_7, s_7) \rightarrow \\
\quad\quad\quad\quad\quad\quad\quad\quad \exists y_8[\mathit{cross}(s_7, y_8, s_8) \wedge \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \forall x_9[\mathit{circle}(s_8, x_9, s_9) \rightarrow \neg \mathit{win}(s_9)]]]]]]]]]
\end{array}
$$

$x_i, y_i$ plays (4 bits each)

$s_i$ configurations

($9 \times 3$ bits each)

- explicit model checking [ClarkeEmerson’82], [Holzmann’91]
  - program presented symbolically (no transition matrix)
  - traversed state space represented explicitly
  - e.g. reached states are explicitly saved bit for bit in hash table
  
  $\Rightarrow$ State Explosion Problem (state space exponential in program size)

- symbolic model checking [McMillan Thesis’93], [CoudertMadre’89]
  - use symbolic representations for sets of states
  - originally with Binary Decision Diagrams [Bryant’86]
  - Bounded Model Checking using SAT [BiereCimattiClarkeZhu’99]

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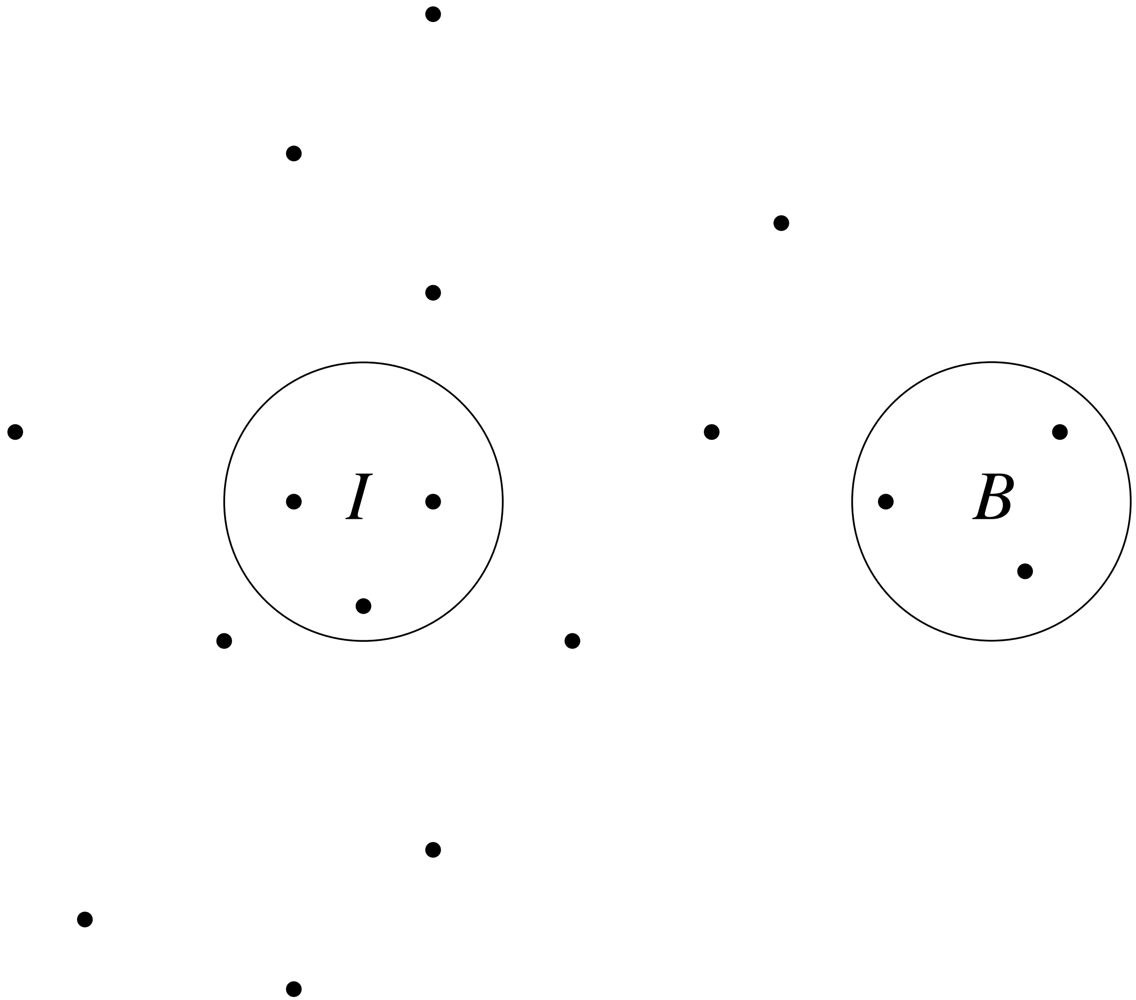

# Forward Fixpoint Algorithm: Step 1



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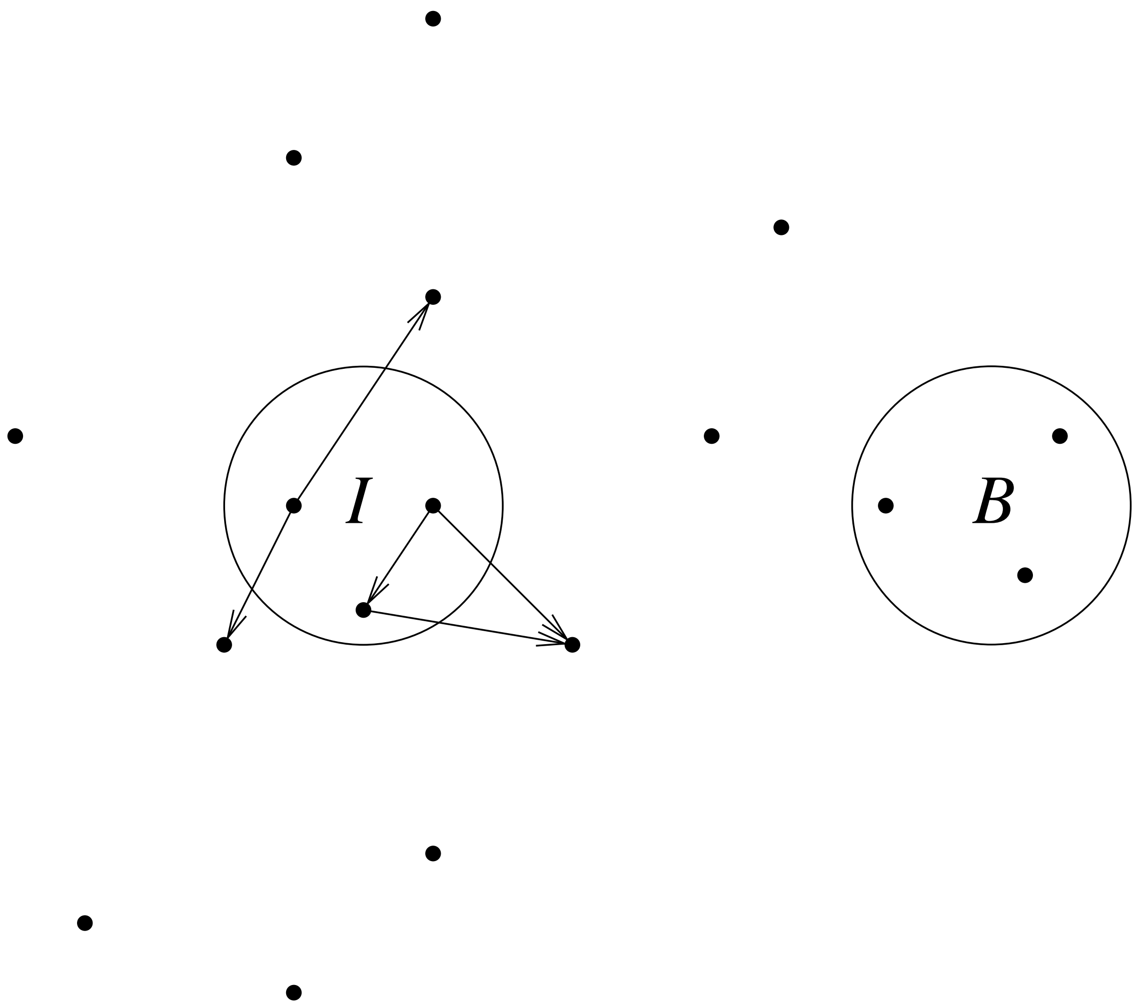

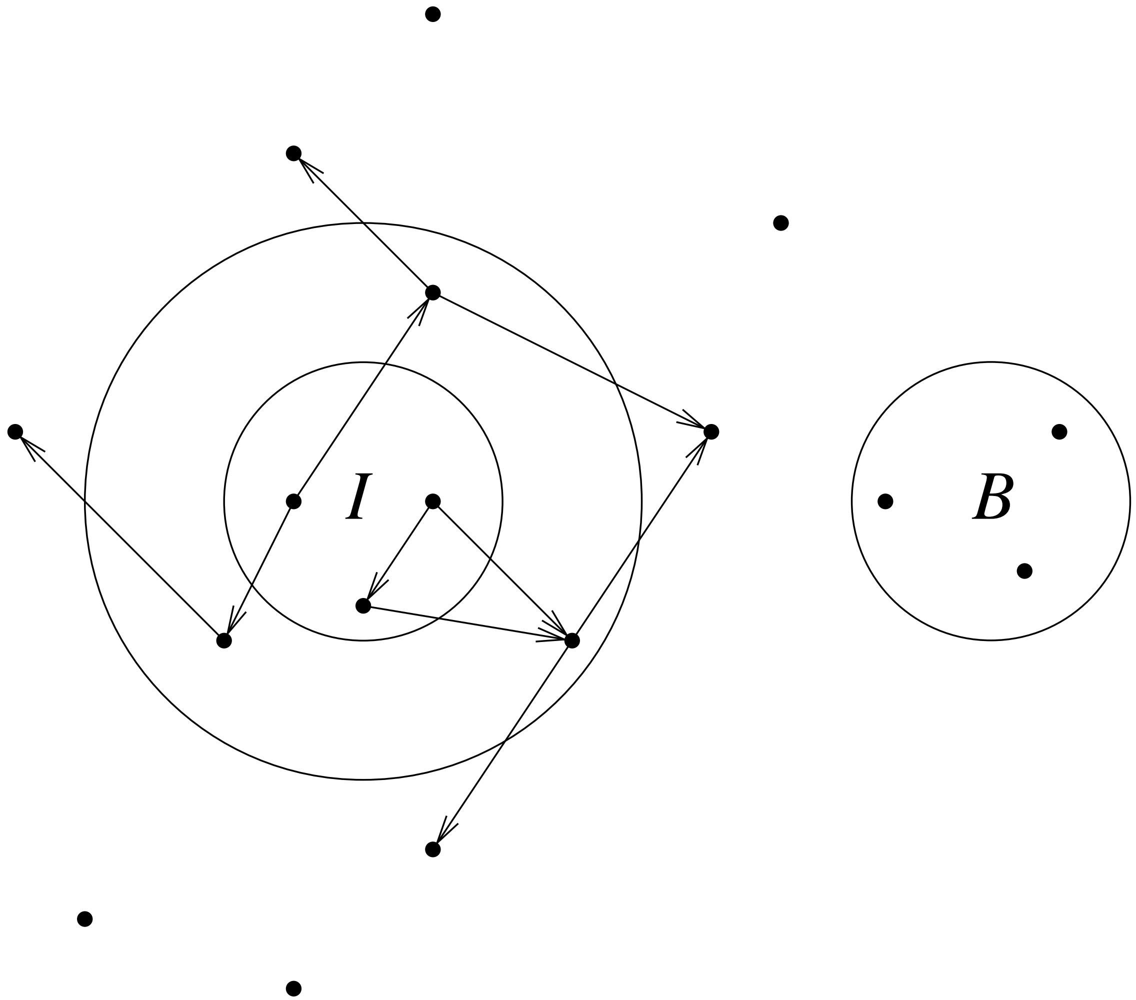

I
B

# Forward Fixpoint Algorithm: Step 3



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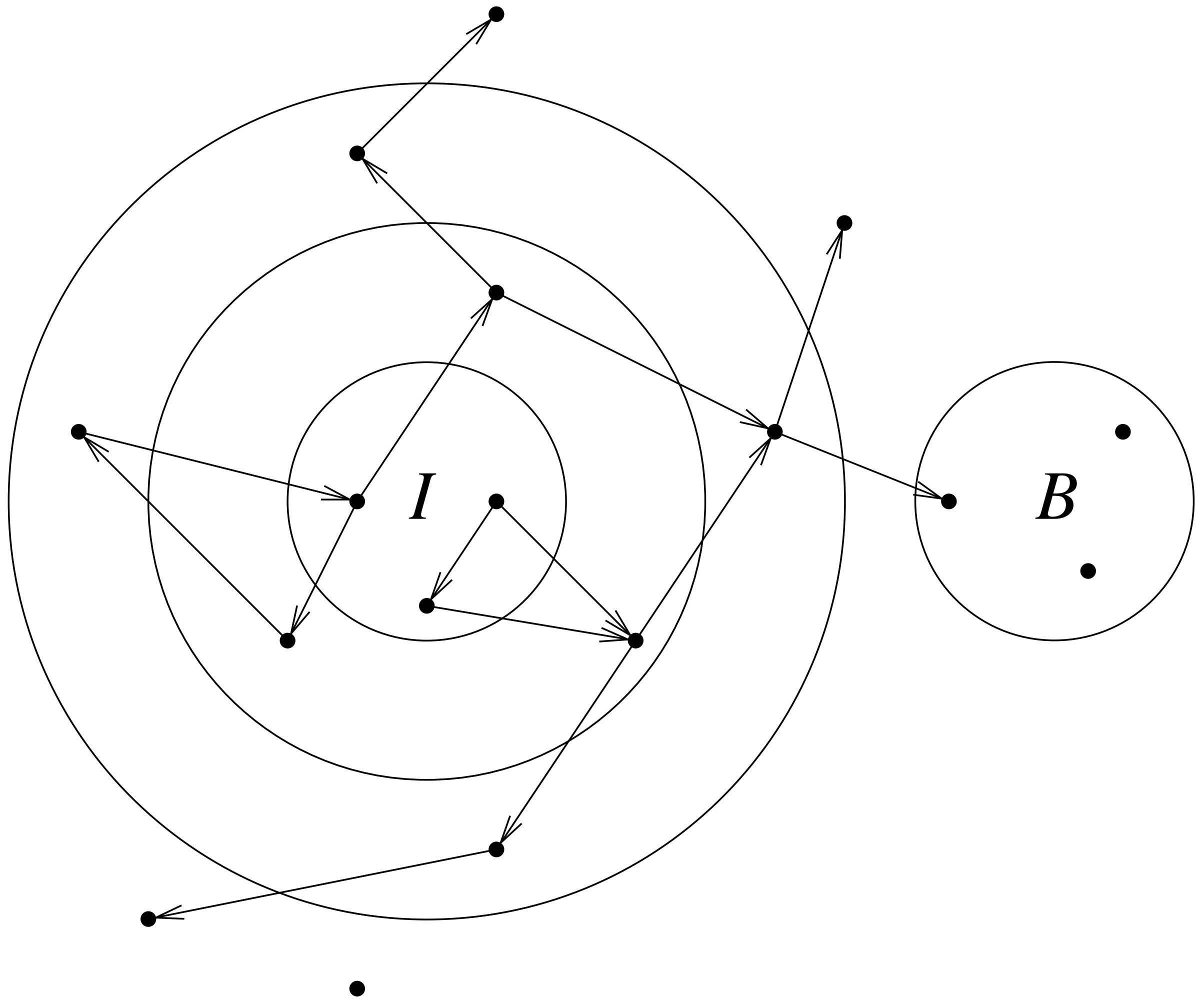

# Forward Fixpoint Algorithm: Bad State Reached



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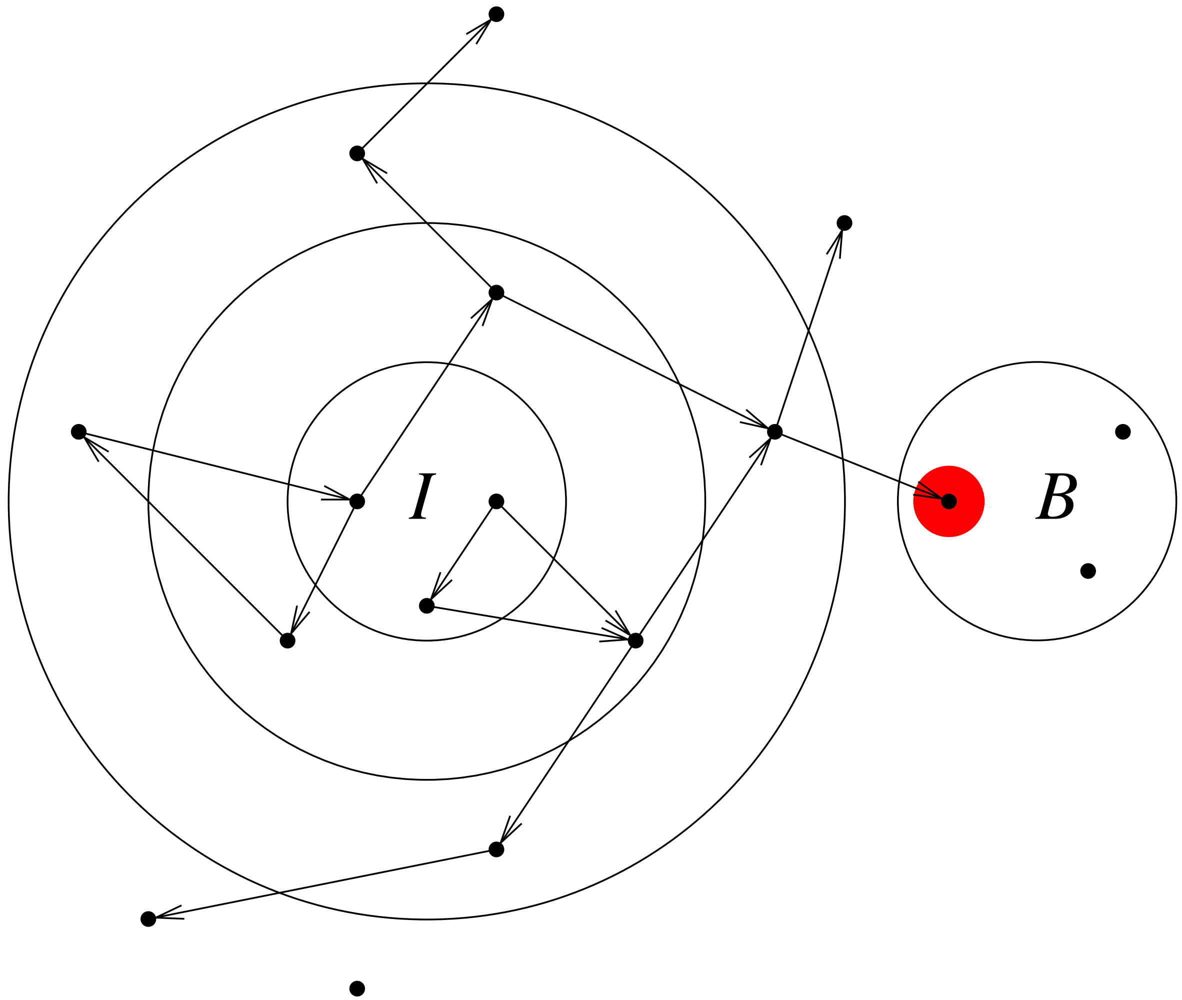

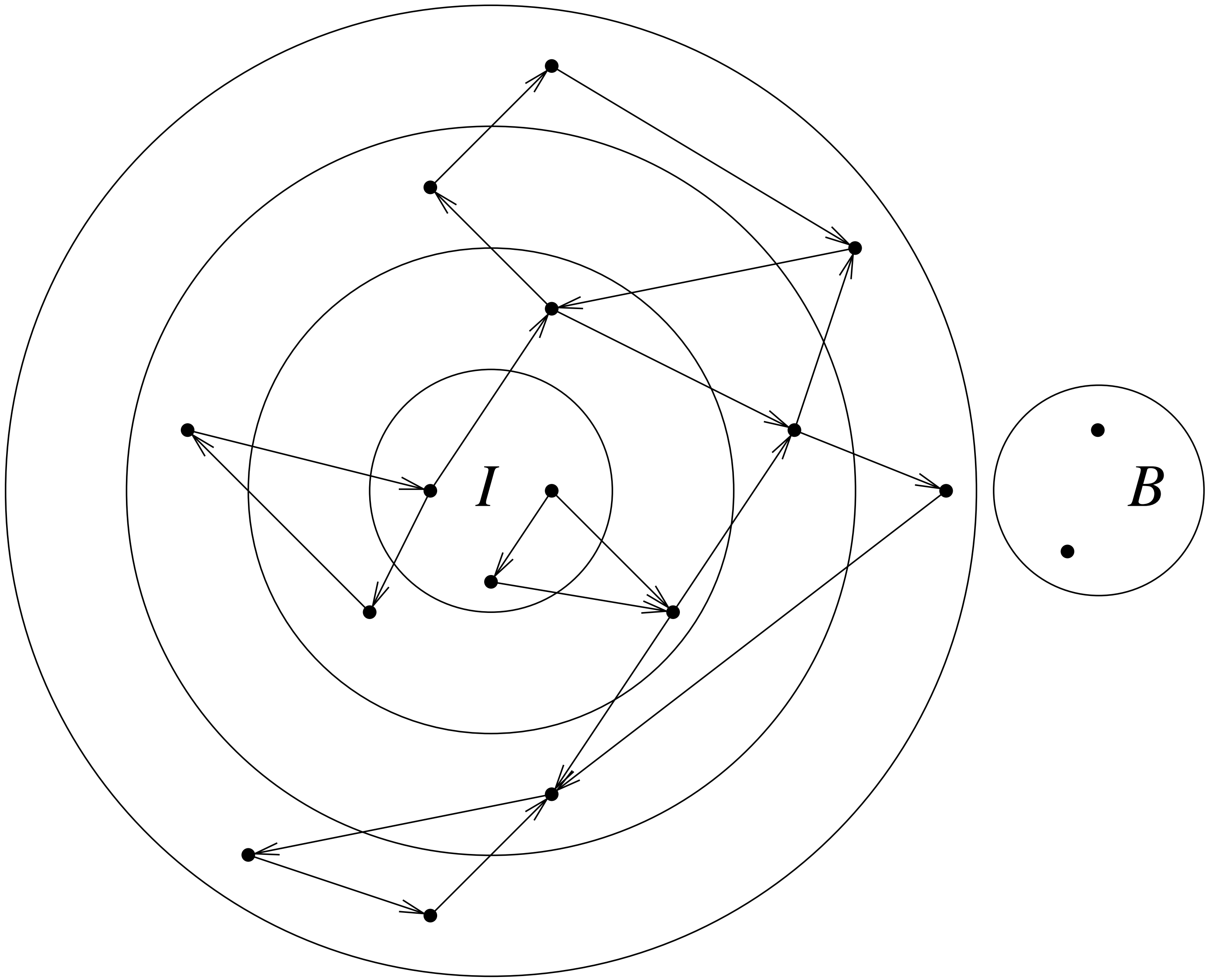
I
B

# Forward Least Fixpoint Algorithm for Model Checking Safety

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initial states $I$, transition relation $T$, bad states $B$

<u>model-check</u>$^{\mu}_{\text{forward}}$ $(I, T, B)$
  $S_C = \emptyset$; $S_N = I$;
  **while** $S_C \neq S_N$ **do**
    **if** $B \cap S_N \neq \emptyset$ **then**
      **return** "found error trace to bad states";
    $S_C = S_N$;
    $S_N = S_C \cup Img(S_C)$;
  **done**;
  **return** "no bad state reachable";

symbolic model checking represents set of states in this BFS symbolically

- BDDs are canonical representation for boolean functions

  – states encoded as bit vectors $\in \mathbb{B}^n$

  – set of states $S \subseteq \mathbb{B}^n$ as BDDs for characteristic function $f_S \colon \mathbb{B}^n \rightarrow \mathbb{B}$

$$f_S(s) = 1 \quad \Leftrightarrow \quad s \in S$$

- for all *set operations* there are linear BDD operations

  – except for $Img$ which is exponential (often also in practice)

$$s \in Img(f) \quad \Leftrightarrow \quad \exists\, t \in \mathbb{B}^n [f(s) \wedge T(s,t)]$$

- variable ordering has strong influence on size of BDDs

- conjunctive partitioning of transition relation is a must

# Unrolling of Forward Least Fixpoint Algorithm



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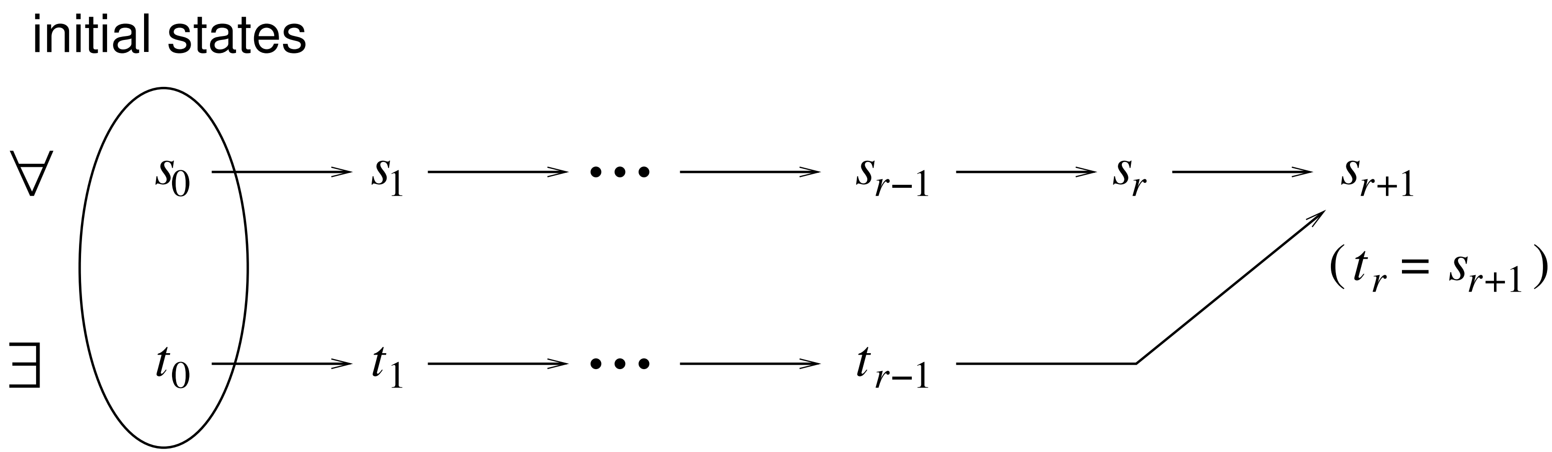
| | | |
|---|---|---|
| 0: continue? | $S_C^0 \neq S_N^0$ | $\exists s_0[I(s_0)]$ |
| 0: terminate? | $S_C^0 = S_N^0$ | $\forall s_0[\neg I(s_0)]$ |
| 0: bad state? | $B \cap S_N^0 \neq \emptyset$ | $\exists s_0[I(s_0) \wedge B(s_0)]$ |
| 1: continue? | $S_C^1 \neq S_N^1$ | $\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge \neg I(s_1)]$ |
| 1: terminate? | $S_C^1 = S_N^1$ | $\forall s_0, s_1[I(s_0) \wedge T(s_0, s_1) \rightarrow I(s_1)]$ |
| 1: bad state? | $B \cap S_N^1 \neq \emptyset$ | $\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge B(s_1)]$ |
| 2: continue? | $S_C^2 \neq S_N^2$ | $\exists s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge$ $\neg(I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)])]$ |
| 2: terminate? | $S_C^2 = S_N^2$ | $\forall s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \rightarrow$ $I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]]$ |
| 2: bad state? | $B \cap S_N^1 \neq \emptyset$ | $\exists s_0, s_1, s_2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge B(s_2)]$ |

$$\forall s_0, \ldots, s_{r+1} [\, I(s_0) \wedge \bigwedge_{i=0}^{r} T(s_i, s_{i+1}) \rightarrow$$

$$\exists t_0, \ldots, t_r [\, I(t_0) \;\wedge\; s_{r+1} = t_r \;\wedge\; \bigwedge_{i=0}^{r-1} (t_i = t_{i+1} \;\vee\; T(t_i, t_{i+1}))]]$$




(we allow $t_{i+1}$ to be identical to $t_i$ in the lower path)

**radius** is smallest $r$ for which formula is true

# Radius Example



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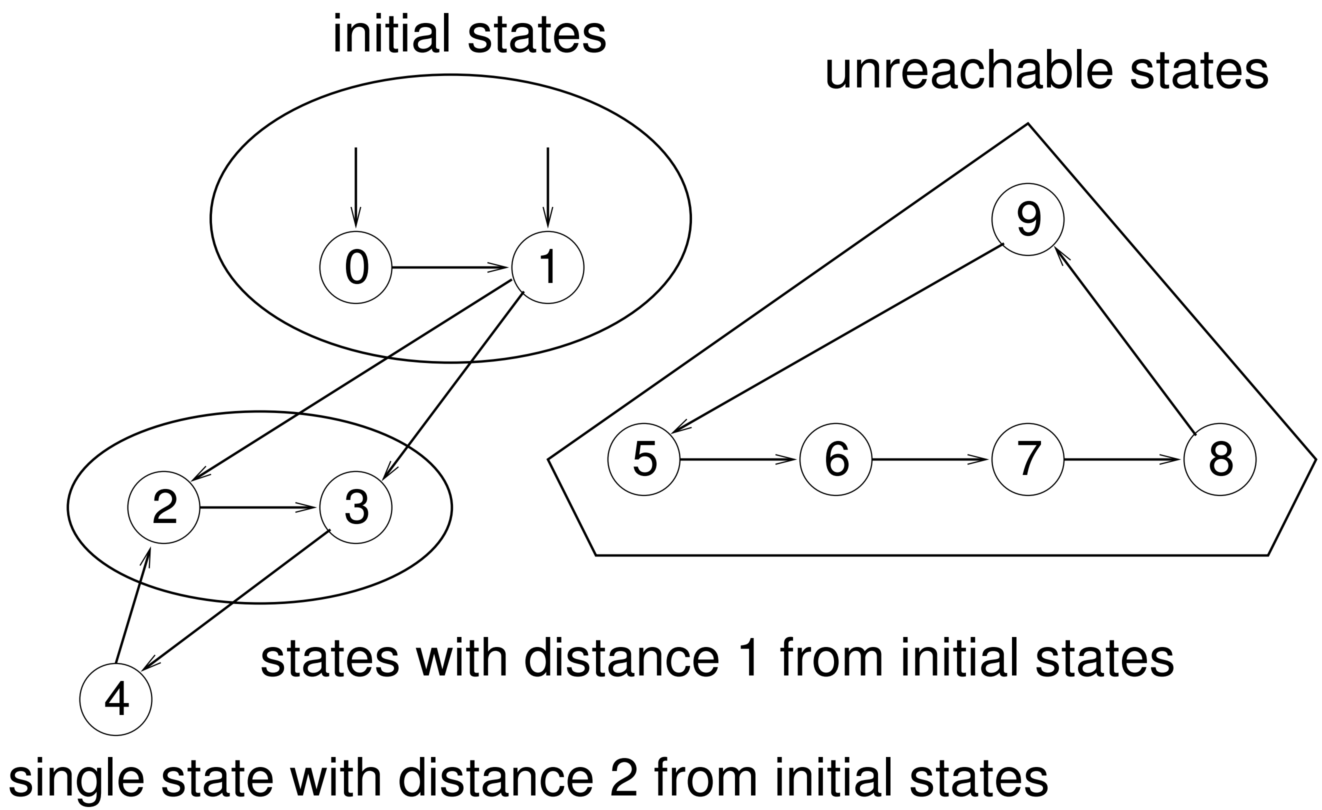

- checking $S_C \neq S_N$ in 2nd iteration results in QBF decision problem

$$\forall s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \rightarrow I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]]$$

- not **eliminating quantifiers** results in QBF with one alternation
  - checking whether bad state is reached only needs SAT
  - number iterations bounded by radius $r = O(2^n)$

- so why not forget about termination and concentrate on bug finding?

$\Rightarrow$ Bounded Model Checking (BMC)

# BMC Part of Fixpoint Algorithm



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~~0: continue? $S_C^0 \neq S_N^0$ $\exists s_0[I(s_0)]$~~

~~0: terminate? $S_C^0 = S_N^0$ $\forall s_0[\neg I(s_0)]$~~

0: bad state? $B \cap S_N^0 \neq \emptyset$ $\exists s_0[I(s_0) \wedge B(s_0)]$

~~1: continue? $S_C^1 \neq S_N^1$ $\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge \neg I(s_1)]$~~

~~1: terminate? $S_C^1 = S_N^1$ $\forall s_0, s_1[I(s_0) \wedge T(s_0, s_1) \rightarrow I(s_1)]$~~

1: bad state? $B \cap S_N^1 \neq \emptyset$ $\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge B(s_1)]$

~~2: continue? $S_C^2 \neq S_N^2$ $\exists s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge$~~
$\neg(I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)])]$

~~2: terminate? $S_C^2 = S_N^2$ $\forall s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \rightarrow$~~
~~$I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]]$~~

2: bad state? $B \cap S_N^1 \neq \emptyset$ $\exists s_0, s_1, s_2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge B(s_2)]$

# Bounded Model Checking (BMC)



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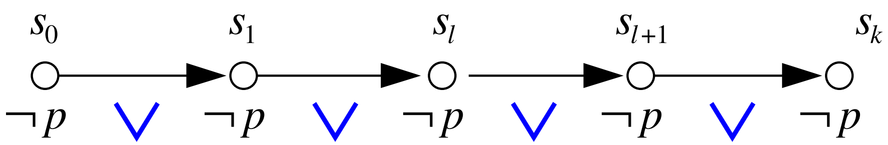
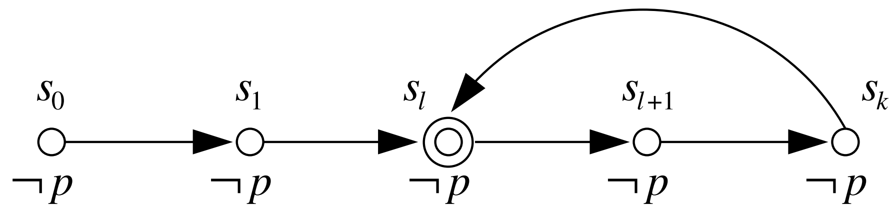
[BiereCimattiClarkeZhu TACAS'99]

- look only for counter example made of $k$ states (the bound)




or




- simple for safety properties $\mathbf{G}p$ (e.g. $p = \neg B$)

$$I(s_0) \wedge (\bigwedge_{i=0}^{k-1} T(s_i, s_{i+1})) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$$

- harder for liveness properties $\mathbf{F}p$

$$I(s_0) \wedge (\bigwedge_{i=0}^{k-1} T(s_i, s_{i+1})) \wedge (\bigvee_{l=0}^{k} T(s_k, s_l)) \wedge \bigwedge_{i=0}^{k} \neg p(s_i)$$

- increase in efficiency of SAT solvers [zchaff]

- SAT more robust than BDDs in bug finding

  (shallow bugs are easily reached by explicit model checking or testing)

- better unbounded but still SAT based model checking algorithms

  - $k$-induction [SinghSheeranStålmarck'00]

  - interpolation [McMillan'03]

- 3rd Intl. Workshop on Bounded Model Checking (BMC'05)

  (11. July, Edinburgh, Scotland)

- other logics beside LTL and better encodings

**Transitive Closure**

$$T^* \equiv T^{2^n}$$

**Standard Linear Unfolding**

$$T^{i+1}(s,t) \equiv \exists m[T^i(s,m) \wedge T(m,t)]$$

**Iterative Squaring via Copying**

$$T^{2\cdot i}(s,t) \equiv \exists m[T^i(s,m) \wedge T^i(m,t)]$$

**Non Copying Iterative Squaring**

$$T^{2\cdot i}(s,t) \equiv \exists m[\forall c[\exists l,r[(c \rightarrow (l,r) = (s,m)) \wedge (\overline{c} \rightarrow (l,r) = (m,t)) \wedge T^i(l,r)]]]$$

```
dpll-sat(Assignment S)          [DavisLogemannLoveland62]
    boolean-constraint-propagation()
    if contains-empty-clause() then return false
    if no-clause-left() then return true
    v := next-unassigned-variable()
    return  dpll-sat(S ∪ {v ↦ false}) ∨ dpll-sat(S ∪ {v ↦ true})
```

```
dpll-qbf(Assignment S)          [CadoliGiovanardiSchaerf98]
    boolean-constraint-propagation()
    if contains-empty-clause() then return false
    if no-clause-left() then return true
    v := next- outermost -unassigned-variable()
    @ := is-existential(v) ? ∨ : ∧
    return  dpll-sat(S ∪ {v ↦ false}) @ dpll-sat(S ∪ {v ↦ true})
```

Why is QBF harder than SAT?

$$\models \quad \forall x \,.\, \exists y \,.\, (x \leftrightarrow y)$$

$$\not\models \quad \exists y \,.\, \forall x \,.\, (x \leftrightarrow y)$$

Decision order matters!

- most implementations DPLL alike: [Cadoli. . .98][Rintanen01]
  - **learning** was added [Giunchiglia. . .01] [Letz01] [ZhangMalik02]
  - top-down: split on variables from the outside to the inside
- multiple quantifier elimination procedures:
  - **enumeration** [PlaistedBiereZhu03] [McMillan02]
  - **expansion** [Aziz-Abdulla. . .00] [WilliamsBiere. . .00] [AyariBasin02]
  - bottom-up: eliminate variables from the inside to the outside
- **q-resolution** [KleineBüning. . .95], with **expansion** [Biere04]
- symbolic representations [PanVardi04] [Benedetti05] BDDs

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- **collect** variables in scopes, **order** scopes according to nesting depth:

$$\underbrace{\exists\, a,b,c,d.}_{\text{scope 0}} \quad \underbrace{\forall x,y,z.}_{\text{scope 1}} \quad \underbrace{\exists\, r,s,t.}_{\text{scope 2}} \quad (c \vee d)(a \vee \overline{c} \vee \overline{x} \vee y)(\overline{a} \vee x \vee s)(t \vee \ldots)\ \cdots$$

**attach** clauses to the scope of its innermost variables

- **remove** innermost univ. literals in clauses attached to univ. scopes:

$$(a \vee \overline{c} \vee \overline{x} \vee y) \quad \text{simplifies to} \quad (a \vee \overline{c})$$

- q-resolution = resolution + forall reduction [KleineBüning...95]

- all clauses are forall reduced

  $\Rightarrow$ innermost scope is always existential

  $\Rightarrow$ no clauses attached to universal scopes

- normalized structure of quantified CNF:

$$\Omega(S_1)\, S_1 \,.\quad \Omega(S_2)\, S_2 \,.\quad \ldots \quad \forall S_{m-1} \,.\quad \exists\, S_m \,.\quad f \wedge g \qquad m \geq 2$$

$$\begin{aligned} f &\equiv \text{clauses of scope} \quad S_m \\ g &\equiv \text{clauses of outer scopes} \quad S_i, \quad i < m-1 \\ S_\exists &\equiv S_m \\ S_\forall &\equiv S_{m-1} \end{aligned}$$

# Resolve and Expand Algorithm

resolve-and-expand()

**forever**

simplify()

**if** contains-empty-clause() **then return** *false*

**if** no-clause-left() **then return** *true*

**if** is-propositional() **then return** sat-solve($\emptyset$)

$v$ := schedule-cheapest-to-eliminate-variable()

**if** is-existential($v$) **then** resolve($v$)

**if** is-universal($v$) **then** expand($v$)

**original** clauses in which $v$ or $\neg v$ occurs:

$\neg r \vee v$

$s \vee v$

$x \vee y \vee v$

$\otimes$

$\neg v \vee r$

$\neg v \vee \neg x \vee \neg y \vee r$

**add** forall reduced non-trivial resolvents:

$(s \vee r)$, $(x \vee y \vee r)$, and $(s \vee \neg x \vee \neg y \vee r)$

**remove** original clauses

# Expand

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one-to-one mapping of variables: $u \in S_\exists$ mapped to $u' \in S'_\exists$

before expansion:

$$\Omega(S_1)\,S_1\,.\quad \Omega(S_2)\,S_2\,.\quad \ldots\quad \forall S_\forall\qquad .\quad \exists\,S_\exists\qquad .\qquad f \wedge g$$

after expansion:

$$\Omega(S_1)\,S_1\,.\quad \Omega(S_2)\,S_2\,.\quad \ldots\quad \forall(S_\forall - \{v\}).\quad \exists(S_\exists \cup S'_\exists).\quad f\{v/0\} \wedge f'\{v/1\} \wedge g$$

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- elimination cost: number of expected added literals

$$
\begin{aligned}
o(l) &\equiv \text{number of clauses with literal } l \\
s(l) &\equiv \text{sum of lengths of clauses with literal } l \\
s(S) &\equiv \text{sum lengths of clauses with scope } S
\end{aligned}
$$

- expansion cost: $\mathrm{s}(\mathrm{S}_\exists) \quad - \quad \Big(\mathrm{s}(\mathrm{v}) + \mathrm{s}(\neg\mathrm{v}) + \mathrm{o}(\mathrm{v}) + \mathrm{o}(\neg\mathrm{v})\Big)$

- resolution cost:

$$\mathrm{o}(\neg\mathrm{v}) \cdot \Big(\mathrm{s}(\mathrm{v}) - \mathrm{o}(\mathrm{v})\Big) + \mathrm{o}(\mathrm{v}) \cdot \Big(\mathrm{s}(\neg\mathrm{v}) - \mathrm{o}(\neg\mathrm{v})\Big) - \Big(\mathrm{s}(\mathrm{v}) + \mathrm{s}(\neg\mathrm{v})\Big)$$

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| | benchmark family | #inst | **decide** | **qube** | **semprop** | *expand* | **quantor** |
|---|---|---|---|---|---|---|---|
| 1 | adder* | 16 | 2 | 2 | 2 | 1 | **3** |
| 2 | Adder2* | 14 | 2 | 2 | 2 | 2 | **3** |
| 3 | C[0-9]* | 27 | 2 | 3 | 2 | 3 | **4** |
| 4 | CHAIN* | 11 | 10 | 7 | **11** | 4 | **11** |
| 5 | comp* | 5 | 4 | 4 | **5** | **5** | **5** |
| 6 | flip* | 7 | 6 | **7** | **7** | **7** | **7** |
| 7 | impl* | 16 | 12 | **16** | **16** | **16** | **16** |
| 8 | k* | 171 | 77 | 91 | 97 | 60 | **108** |
| 9 | mutex* | 2 | 1 | **2** | **2** | **2** | **2** |
| 10 | robots* | 48 | 0 | **36** | **36** | 15 | 24 |
| 11 | term1* | 4 | 2 | **3** | **3** | 1 | **3** |
| 12 | toilet* | 260 | 187 | **260** | **260** | 259 | 259 |
| 13 | TOILET* | 8 | **8** | 6 | **8** | **8** | **8** |
| 14 | tree* | 12 | 10 | **12** | **12** | 8 | **12** |
| #(among best in family) | | | **1** | **7** | **10** | **5** | **12** |
| #(single best in family) | | | **0** | **0** | **0** | **0** | **4** |

(families with no difference and two actually random families removed)

- rectification problems (actually a synthesis problem)

$$\exists p[\forall i[g(i,p) = s(i)]]$$

with parameters $p$, inputs $i$, generic circuit $g$, and specification $s$

- games, open systems, non-deterministic planning applications?

- model checking
  - termination check as in classical (BDD based) model checking

    (only one alternation)
  - acceleration as in PSPACE completeness for QBF proof

    (at most linear number of alternations in number of state bits $n$)

# QBF in Software Verification



[CookKröningSharygina – SMC'05]

- model: asynchronous boolean programs
  parallel version of those used in SLAM, BLAST or MAGIC

- symbolic representation of set of states
  - related work uses BDDs
  - [CookKröningSharygina] boolean formulas

- termination check for reachability (partially explicit)
  - trivial with BDDs as symbolic representation
  - QBF decision procedure for boolean formulas $\Leftarrow$ [quantor]

- SAT/QBF version seems to scale much better than BDDs

## Summary

- applications fuel interest in SAT
  - incredible capacity increase in recent years
    (instances with thousands or million variable are regularly solved)
  - SAT solver competition affiliated to SAT conference
  - SAT is becoming a core verification technology

- QBF is catching up
  - solvers are getting better
  - new applications
  - richer structure

# Solved Hard Instances of SAT'03 QBF Evaluation: QUANTOR

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| | hard instance | time | space | ∀ | ∃ | units | pure | subsu. | subst. | ∀red. |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Adder2-6-s | 29.6 | 19.7 | 90 | 13732 | 126 | 13282 | 174081 | 0 | 37268 |
| 2 | adder-4-sat | 0.2 | 2.8 | 42 | 1618 | 0 | 884 | 6487 | 0 | 960 |
| 3 | adder-6-sat | 36.6 | 22.7 | 90 | 13926 | 0 | 7290 | 197091 | 0 | 54174 |
| 4 | C49*1.*_0_0* | 27.9 | 13.3 | 1 | 579 | 0 | 0 | 48 | 84 | 0 |
| 5 | C5*1.*_0_0* | 56.2 | 16.0 | 2 | 2288 | 10 | 0 | 4552 | 2494 | 0 |
| 6 | k_path_n-15 | 0.1 | 0.8 | 32 | 977 | 66 | 82 | 2369 | 2 | 547 |
| 7 | k_path_n-16 | 0.1 | 0.8 | 34 | 1042 | 69 | 85 | 2567 | 2 | 597 |
| 8 | k_path_n-17 | 0.1 | 0.9 | 36 | 1087 | 72 | 100 | 3020 | 2 | 639 |
| 9 | k_path_n-18 | 0.1 | 0.9 | 36 | 1146 | 76 | 106 | 3242 | 2 | 725 |
| 10 | k_path_n-20 | 0.1 | 0.9 | 38 | 1240 | 84 | 149 | 3967 | 2 | 855 |
| 11 | k_path_n-21 | 0.1 | 1.0 | 40 | 1318 | 84 | 130 | 4470 | 2 | 909 |
| 12 | k_t4p_n-7 | 15.5 | 105.8 | 43 | 88145 | 138 | 58674 | 760844 | 8 | 215 |
| 13 | k_t4p_p-8 | 5.8 | 178.6 | 29 | 12798 | 206 | 5012 | 85911 | 4 | 138 |
| 14 | k_t4p_p-9 | 0.3 | 4.5 | 32 | 4179 | 137 | 1389 | 23344 | 10 | 142 |
| 15 | k_t4p_p-10 | 27.9 | 152.9 | 35 | 130136 | 193 | 63876 | 938973 | 4 | 137 |
| 16 | k_t4p_p-11 | 86.0 | 471.5 | 38 | 196785 | 204 | 79547 | 1499430 | 4 | 140 |
| 17 | k_t4p_p-15 | 84.6 | 354.7 | 50 | 240892 | 169 | 181676 | 1336774 | 9 | 226 |
| 18 | k_t4p_p-20 | 3.6 | 16.1 | 65 | 27388 | 182 | 21306 | 197273 | 11 | 325 |

time in seconds, space in MB

- specific workshop: Satisfiability Modulo Theories (SMT'05)

- examples

  - processor verification [BurchDill CAV'94] [VelevBryant JSC'03]

  - translation validation [PnueliStrichmanSiegel'98]

- eager approach: translate into SAT

- lazy approach

  - augment SAT solver to handle non-propositional constraints

  - in each branch: SAT part satisfiable, check non-propositional theory

# Examples for Using SAT in Software Verification



A. Biere, FMV, JKU Linz

- [JacksonVaziri ISSTA'00] Alloy
  - bounded model checking of OO modelling language Alloy
  - checks properties of symbolic simulations with bounded heap size
- [KroeningClarkeYorav DAC03] CBMC
  - targets equivalence checking of hardware models
  - bounded model checking of C resp. Verilog programs
- [XieAiken POPL'05] Saturn
  - LINT for lock usage in large C programs (latest Linux kernel)
  - neither sound nor complete, but 179 bugs out of 300 warnings