ALGEBRA, PROOFS AND MULTIPLIERS

Armin Biere joint work with Daniela Kaufmann and Manuel Kauers

28th International Workshop on Logic & Synthesis

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FWF Der Wissenschaftsfonds.

RiSE Rigorous Systems Engineering

JKU JOHANNES KEPLER UNIVERSITÄT LINZ

Multiplication

$$13 \cdot 15$$

Multiplication

$$
\begin{array}{rrcrr}
1 & 3 & \cdot & 1 & 5 \\
\hline
 & & & 6 & 5 \\
 & & & & \\
\hline
\end{array}
$$

Multiplication

$$
\begin{array}{rrcrr}
1 & 3 & \cdot & 1 & 5 \\
\hline
 & & & 6 & 5 \\
 & & 1 & 3 & \\
\hline
\end{array}
$$

Multiplication

$$
\begin{array}{rrcrr}
1 & 3 & \cdot & 1 & 5 \\
\hline
 & & & 6 & 5 \\
 & & 1 & 3_{0} & \\
\hline
 & & & & 5
\end{array}
$$

Multiplication

$$
\begin{array}{rrcrr}
1 & 3 & \cdot & 1 & 5 \\
\hline
 & & & 6 & 5 \\
 & & 1_{0} & 3_{0} & \\
\hline
 & & & 9 & 5
\end{array}
$$

Multiplication

$$
\begin{array}{rrcrr}
1 & 3 & \cdot & 1 & 5 \\
\hline
 & & & 6 & 5 \\
 & & 1 & 3 & \\
 & 0 & 0 & 0 & \\
\hline
 & & 1 & 9 & 5
\end{array}
$$

Binary multiplication

$$
\begin{array}{cccccccccc}
 & 1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & & & & & \\
\hline
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
\hline
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccc|cccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & & & & & & \\
 & & & & & & & & \\
\hline
\end{array}
$$

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Binary multiplication

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\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & & & & & & & \\
\hline
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1 & 1 & 0 & 1 & & & \\
\hline
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1 & 1 & 0 & 1 & & & \\
\hline
 & & & & & & & & 1
\end{array}
$$

$13 \cdot 15 = 195$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1 & 1 & 0 & 1 & & & \\
 & & & & & & 0 & 0 & \\
\hline
 & & & & & & & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1 & 1 & 0 & 1 & & & \\
 & & & & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & \\
\hline
 & & & & & & 0 & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1 & 1 & 0 & 1 & & & \\
 & & & & {\scriptstyle 2} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & \\
\hline
 & & & & & 0 & 0 & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1 & 1 & 0 & 1 & & & \\
 & & & {\scriptstyle 2} & {\scriptstyle 2} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & \\
\hline
 & & & & 0 & 0 & 0 & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1_{2} & 1_{2} & 0_{2} & 1_{1} & 0 & 0 & \\
\hline
 & & & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1_{1} & 1_{2} & 0_{2} & 1_{2} & 0_{1} & 0 & \\
\hline
 & & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

Binary multiplication

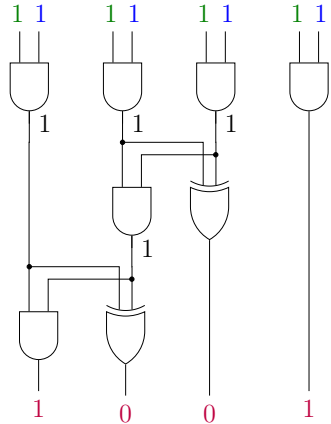
$$
\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 0 & 1 & \\
 & & & 1 & 1 & 0 & 1 & & \\
 & & 1_{1} & 1_{2} & 0_{2} & 1_{1} & 0 & 0 & \\
\hline
 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

$$13 \cdot 15 = 195$$

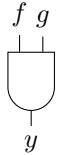
Example: 2 Bit - Binary Multiplication

$$
\begin{array}{cccc}
1 & 1 & \cdot & 1 & 1 \\
\hline
 & & & 1 & 1 \\
 & & 1 & 1 & \\
 & 1 & 1 & 0 & \\
\hline
1 & 0 & 0 & 1 &
\end{array}
$$

$3 \cdot 3 = 9$



Example: 2 Bit - Binary Multiplication

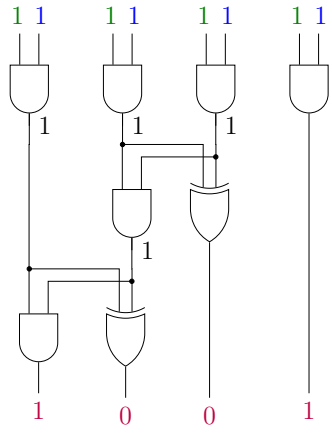


AND-Gate

f	g	y
0	0	0
0	1	0
1	0	0
1	1	1

XOR-Gate

f	g	y
0	0	0
0	1	1
1	0	1
1	1	0



Motivation & Solving Techniques

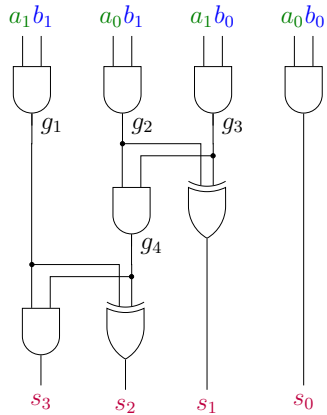
Given: Gate-level multiplier for fixed bit-width n.

Question: For all possible $a_i, b_i \in \mathbb{B}$:

$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$

Solving Techniques

- SAT using CNF encoding
- Binary Moment Diagrams (BMD)
- Algebraic reasoning



Previous Work

- **SAT using CNF encoding**
  - A. Biere. **Weakness** of CDCL solvers. SAT Solving Workshop, 2016.
  - P. Beame and V. Liew. **Towards verifying** nonlinear integer arithmetic. In CAV, 2017.
- **Binary moment diagrams**
  - Y.-A. Chen and R.E. Bryant. Verification of arithmetic circuits with **binary moment diagrams**. In DAC, 1995.
- **Algebraic reasoning**
  - O. Wienand, M. Wedler, D. Stoffel, W. Kunz, and G.-M. Greuel. An algebraic approach for proving data correctness in **arithmetic data paths**. In CAV, 2008.
  - J. Lv, P. Kalla, and F. Enescu. Efficient Gröbner basis reductions for formal verification of **Galois field arithmetic circuits**. In IEEE TCAD, 2013.
  - C. Yu, W. Brown, D. Liu, A. Rossi, and M. Ciesielski. Formal verification of arithmetic circuits by **function extraction**. In IEEE TCAD, 2016.
  - A.A.R. Sayed-Ahmed, D. Große, U. Kühne, M. Soeken, and R. Drechsler. Formal verification of integer multipliers by combining **Gröbner basis with logic reduction**. In DATE, 2016.

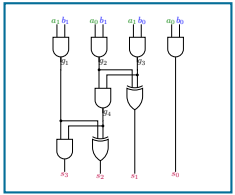
Recent Work

- D. Ritirc, A. Biere, and M. Kauers. Column-wise verification of multipliers using computer algebra. In FMCAD'17.
- D. Ritirc, A. Biere, and M. Kauers. A Practical Polynomial Calculus for Arithmetic Circuit Verification. In 3rd International Workshop on Satisfiability Checking and Symbolic Computation (SC2'18), pages 61–76. CEUR-WS, 2018.
- D. Kaufmann, A. Biere, and M. Kauers. Incremental Column-wise verification of arithmetic circuits using computer algebra. FMSD, Feb 2019
- D. Kaufmann, M. Kauers, A. Biere, and D. Cok. Arithmetic Verification Problems Submitted to the SAT Race 2019. In Proc. of SAT Race 2019, 2019.

- M. Ciesielski, T. Su, A. Yasin, and C. Yu. Understanding Algebraic Rewriting for Arithmetic Circuit Verification: a Bit-Flow Model. IEEE TCAD, pages 1–1, 2019.
- A. Mahzoon, D. Große, and R. Drechsler. PolyCleaner: clean your polynomials before backward rewriting to verify million-gate multipliers. In ICCAD'18.
- A. Mahzoon, D. Große, and R. Drechsler. RevSCA: Using Reverse Engineering to Bring Light into Backward Rewriting for Big and Dirty Multipliers. In DAC'19.

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$
\begin{aligned}
B = \{ & \\
& x - a_0 * b_0, \\
& y - a_1 * b_1, \\
& s_0 - x * y, \\
& \dots \\
\} &
\end{aligned}
$$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - (\sum_{i=0}^{n-1} 2^i a_i)(\sum_{i=0}^{n-1} 2^i b_i)$$

Ideal Membership Test

$= 0$ ✓

$\neq 0$ ✗

Polynomials

$$p = c_1\tau_1 + \ldots + c_m\tau_m \quad \in \quad \mathbb{Q}[X] = \mathbb{Q}[x_1, \ldots, x_n]$$

- $\mathbb{Q}[X]$ is the **ring of polynomials** with variables $X = x_1, \ldots, x_n$ and coefficients in $\mathbb{Q}$.
- A **term** τ_i is a product $x_1^{e_1} \cdots x_n^{e_n}$ with $e_j \geq 0$.
- A **monomial** $c_i\tau_i$ is a constant multiple of a term with $c_i \in \mathbb{Q}$.
- A **polynomial** p is a finite sum of monomials.

Polynomials

$$p = c_1\tau_1 + \ldots + c_m\tau_m \quad \in \quad \mathbb{Q}[X] = \mathbb{Q}[x_1, \ldots, x_n]$$

- We fix a **term order** such that for all terms τ, σ_1, σ_2 we have $x_1^0 \cdots x_n^0 = 1 \leq \tau$ and $\sigma_1 \leq \sigma_2 \Rightarrow \tau\sigma_1 \leq \tau\sigma_2$.
- An order is a **lexicographic term order** if for all $\sigma_1 = x_1^{u_1} \cdots x_n^{u_n}$, $\sigma_2 = x_1^{v_1} \cdots x_n^{v_n}$ we have $\sigma_1 < \sigma_2$ iff there exists an index i with $u_j = v_j$ for all $j < i$, and $u_i < v_i$.

- $\mathrm{lm}(p) = c_1\tau_1$ is the **leading monomial** of p.
- $\mathrm{lt}(p) = \tau_1$ is the **leading term** of p.
- $p - \mathrm{lm}(p)$ is the **tail** of p.

Ideals

Ideal. A nonempty subset $I \subseteq \mathbb{Q}[X]$ is called an ideal if

$$\forall\, p, q \in I : p + q \in I \qquad \text{and} \qquad \forall\, p \in \mathbb{Q}[X]\ \forall\, q \in I : pq \in I$$

Basis. A set $P = \{p_1, \ldots, p_m\} \subseteq \mathbb{Q}[X]$ is called a **basis** of an ideal I if

$$I = \{q_1 p_1 + \cdots + q_m p_m \mid q_1, \ldots, q_m \in \mathbb{Q}[X]\} = \langle P \rangle$$

I is the set of polynomials which become zero, when the elements of P become zero.

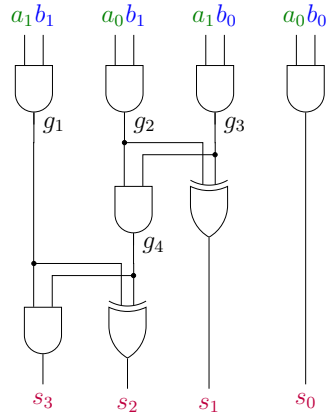
Circuit Polynomials

Gate polynomials.

$$
\begin{aligned}
s_3 = g_1 \wedge g_4 \quad & -s_3 + g_1 g_4, \\
s_2 = g_1 \oplus g_4 \quad & -s_2 + g_1 + g_4 - 2g_1 g_4, \\
g_4 = g_2 \wedge g_3 \quad & -g_4 + g_2 g_3, \\
s_1 = g_2 \oplus g_3 \quad & -s_1 + g_2 + g_3 - 2g_2 g_3, \\
g_1 = a_1 \wedge b_1 \quad & -g_1 + a_1 b_1, \\
g_2 = a_0 \wedge b_1 \quad & -g_2 + a_0 b_1, \\
g_3 = a_1 \wedge b_0 \quad & -g_3 + a_1 b_0, \\
s_0 = a_0 \wedge b_0 \quad & -s_0 + a_0 b_0
\end{aligned}
$$

Input Field polynomials.

$$
\begin{aligned}
a_1, a_0 \in \mathbb{B} \quad & a_1(1-a_1),\ a_0(1-a_0), \\
b_1, b_0 \in \mathbb{B} \quad & b_1(1-b_1),\ b_0(1-b_0)
\end{aligned}
$$

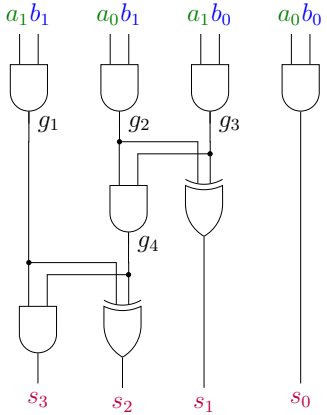


Ideals associated to Circuits

Polynomial Circuit Constraints (PCCs).
A polynomial $p \in \mathbb{Q}[X]$ such that for all

$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \in \{0,1\}^{2n}$$

and resulting values $g_1, \ldots, g_k, s_0, \ldots, s_{2n-1}$ implied by the gates of the circuit C the substitution of these values into p gives zero.

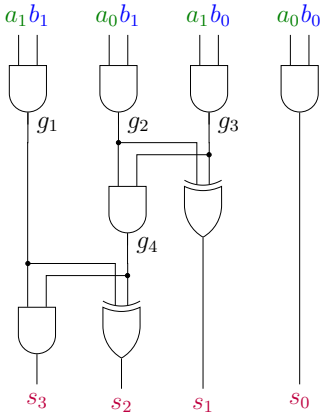


- The set of all PCCs is denoted by $I(C)$.
- $I(C)$ contains all relations of the circuit.
- $I(C)$ is an ideal.

Ideals associated to Circuits

Examples for PCCs:

- $s_0 - a_0 b_0$ and gate
- $a_1^2 - a_1$ a_1 boolean
- $g_2^2 - g_2$ g_2 boolean
- $s_1 g_4$ xor-and constraint
- $\ldots$



Multiplier. A circuit C is called a multiplier if

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right) \quad \in \quad I(C).$$

Ideal Membership Test

Ideal membership problem. Given a polynomial $f \in \mathbb{Q}[X]$ and an ideal $I = \langle g_1, \ldots, g_m \rangle = \langle G \rangle \subseteq \mathbb{Q}[X]$, determine if $f \in I$.

Given arbitrary basis G of I it is not obvious how to solve ideal membership problem.

Lemma (Ideal membership test)

Let $G = \{g_1, \ldots, g_m\} \subseteq \mathbb{Q}[X]$ be a Gröbner basis, and let $f \in \mathbb{Q}[X]$. Then f is contained in the ideal $I = \langle G \rangle$ iff the unique remainder of f with respect to G is zero.

Gröbner basis

- Every ideal $I \subseteq \mathbb{Q}[X]$ has a **Gröbner basis** w.r.t. a fixed term order.

- Construction algorithm by Buchberger which given an arbitrary basis of an ideal I computes a Gröbner basis of it (double exponential)
- Algorithm is based on repeated reduction of so-called S-polynomials (spol).

- A basis G is a Gröbner basis of $I = \langle G \rangle$ if for all $p, q \in G : \mathrm{spol}(p, q)$ reduces to zero.
- **Product criterion.** If $p, q \in \mathbb{Q}[X] \setminus \{0\}$ are such that the leading terms are coprime, i.e., $\mathrm{lcm}(\mathrm{lt}(p), \mathrm{lt}(q)) = \mathrm{lt}(p)\,\mathrm{lt}(q)$, then $\mathrm{spol}(p, q)$ reduces to zero.

Circuit Gröbner basis

We can deduce at least some elements of $I(C)$:

- G = Gate Polynomials + Input Field Polynomials
- Let $J(C) = \langle G \rangle$.
- Lexicographic term order: output variable of a gate is greater than input variables

Theorem

G is a Gröbner basis for $J(C)$.

Proof idea: Application of Buchberger's Product criterion.

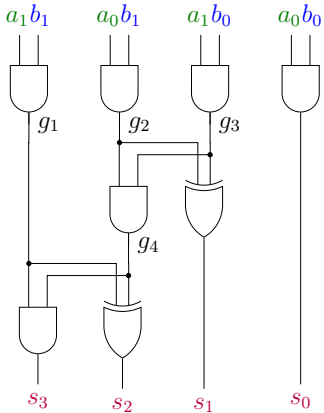
Circuit Polynomials

Gate polynomials.

$$
\begin{aligned}
s_3 = g_1 \wedge g_4 \quad & -s_3 + g_1 g_4, \\
s_2 = g_1 \oplus g_4 \quad & -s_2 - 2g_1 g_4 + g_4 + g_1, \\
g_4 = g_2 \wedge g_3 \quad & -g_4 + g_2 g_3, \\
s_1 = g_2 \oplus g_3 \quad & -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 = a_1 \wedge b_1 \quad & -g_1 + a_1 b_1, \\
g_2 = a_0 \wedge b_1 \quad & -g_2 + a_0 b_1, \\
g_3 = a_1 \wedge b_0 \quad & -g_3 + a_1 b_0, \\
s_0 = a_0 \wedge b_0 \quad & -s_0 + a_0 b_0
\end{aligned}
$$

Input Field polynomials.

$$
\begin{aligned}
a_1, a_0 \in \mathbb{B} \quad & -a_1^2 + a_1,\ -a_0^2 + a_0, \\
b_1, b_0 \in \mathbb{B} \quad & -b_1^2 + b_1,\ -b_0^2 + b_0
\end{aligned}
$$



Soundness and completeness

Theorem (Soundness and completeness)

For all acyclic circuits C, we have $J(C) = I(C)$.

- $J(C) \subset I(C)$: soundness
- $I(C) \subset J(C)$: completeness

Non-Incremental Checking Algorithm

Non-Incremental Checking Algorithm.

Divide polynomial $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of G until no further reduction is possible, then C is a multiplier iff remainder is zero.

Implications:

- Leading term is one variable, division is actually substitution by tail.
- Leading coefficient ± 1 of all gate polynomials, computation stays in $\mathbb{Z}$.
- Still can use rational coefficients $\mathbb{Q}$ (important for Singular).
- Completeness proof allows to derive input assignment if C is incorrect.

Example: 2 Bit - Binary Multiplication

$$
\begin{aligned}
G = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 + g_1 + g_4 - 2g_1 g_4, \\
& -g_4 + g_2 g_3, \\
& -s_1 + g_2 + g_3 - 2g_2 g_3, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Example: 2 Bit - Binary Multiplication

$$
\begin{aligned}
G = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 + g_1 + g_4 - 2g_1 g_4, \\
& -g_4 + g_2 g_3, \\
& -s_1 + g_2 + g_3 - 2g_2 g_3, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Example: 2 Bit - Binary Multiplication

$$
\begin{aligned}
G = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 + g_1 + g_4 - 2 g_1 g_4, \\
& -g_4 + g_2 g_3, \\
& -s_1 + g_2 + g_3 - 2 g_2 g_3, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

$$8 s_3 + 4 s_2 + 2 s_1 + s_0 - 4 a_1 b_1 - 2 a_1 b_0 - 2 a_0 b_1 - a_0 b_0$$

$$8 g_1 g_4 + 4 s_2 + 2 s_1 + s_0 - 4 a_1 b_1 - 2 a_1 b_0 - 2 a_0 b_1 - a_0 b_0$$

$$\begin{aligned} 8 g_1 g_4 + 4 (g_1 + g_4 - 2 g_1 g_4) + 2 s_1 + s_0 \\ - 4 a_1 b_1 - 2 a_1 b_0 - 2 a_0 b_1 - a_0 b_0 \end{aligned}$$

Example: 2 Bit - Binary Multiplication

$$
\begin{aligned}
G = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 + g_1 + g_4 - 2g_1 g_4, \\
& -g_4 + g_2 g_3, \\
& -s_1 + g_2 + g_3 - 2g_2 g_3, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

$$
\begin{aligned}
& 8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
& 8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
& 8g_1 g_4 + 4(g_1 + g_4 - 2g_1 g_4) + 2s_1 + s_0 \\
& \qquad - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
& \vdots \\
& 0
\end{aligned}
$$

Computational Issues

Generally the number of monomials in the intermediate results increases drastically:

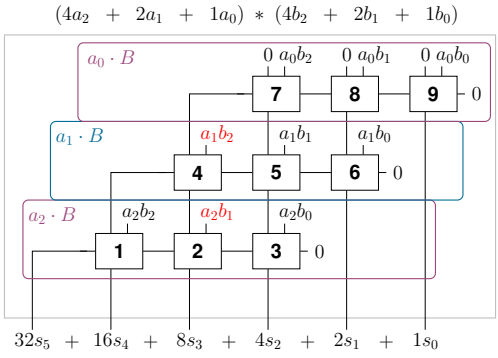
- 8-bit multiplier can not be verified within 20 minutes.

Tailored heuristics become very important:

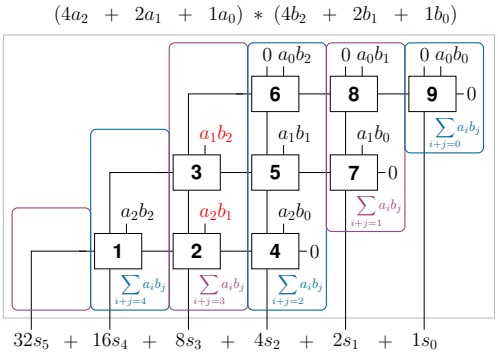
- Choose appropriate term order.
- Divide verification problem into smaller sub-problems.
- Rewrite and thus simplify Gröbner basis G.

Order

Row-Wise



Column-Wise



Slicing

Partial Products. Let $P_k = \sum_{k=i+j} a_i b_j$.

Input Cone. For each output bit s_i we determine its input cone

$$I_i = \{\text{gate } g \mid g \text{ is in input cone of output } s_i\}$$

Slice. Slices S_i are defined as the difference of consecutive cones I_i:

$$S_0 = I_0 \qquad S_{i+1} = I_{i+1} \setminus \bigcup_{j=0}^{i} S_j$$

Sliced Gröbner Bases. Let G_i be the set of gate and input field polynomials in S_i.

Carry Recurrence Relation

Carry Recurrence Relation.
A sequence of $2n+1$ polynomials $C_0, \ldots, C_{2n}$ is called a **carry sequence** if

$$-C_i + 2C_{i+1} + s_i - P_i \in I(C) \quad \text{for all } 0 \leq i < 2n+1.$$

Then $R_i = -C_i + 2C_{i+1} + s_i - P_i$ are the **carry recurrence relations** for $C_0, \ldots, C_{2n}$.

Theorem

Let C be a circuit where all carry recurrence relations are contained in $I(C)$.
Then C is a multiplier, iff $C_0 - 2^{2n} C_{2n} \in I(C)$.

Incremental Algorithm

Incremental Checking Algorithm.

input: Circuit C with sliced Gröbner bases G_i
output: Determine whether C is a multiplier

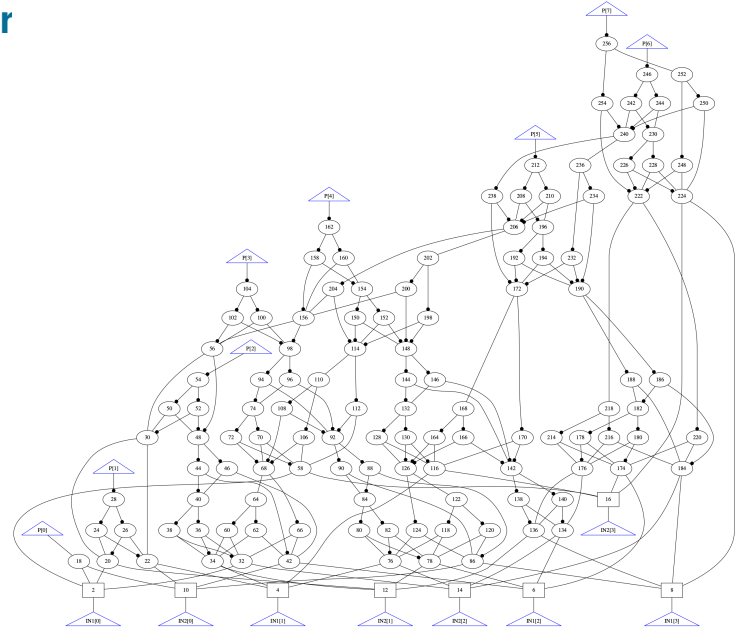
$C_{2n} \leftarrow 0$

for $i \leftarrow 2n-1$ **to** 0

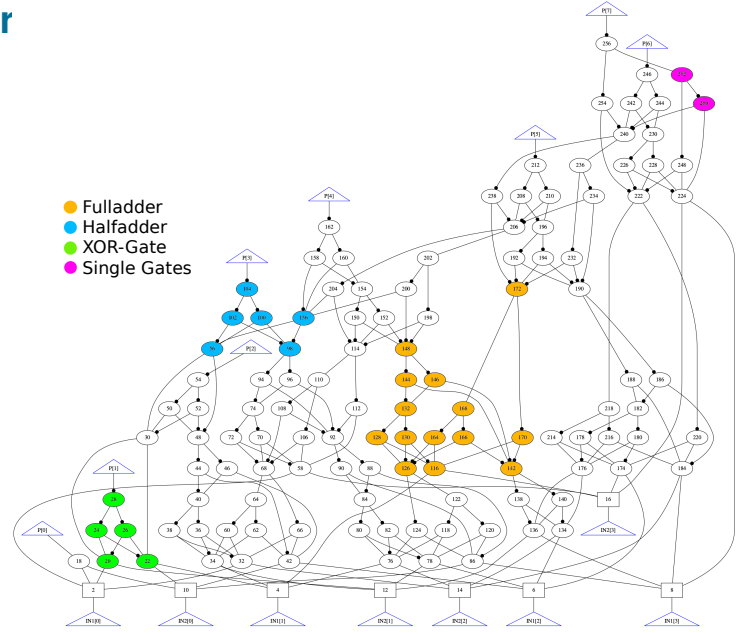
$C_i \leftarrow$ Remainder ($2C_{i+1} + s_i - P_i,\ \ G_i$)

return $C_0 = 0$

Multiplier



Multiplier


Fulladder
Halfadder
XOR-Gate
Single Gates
P[7]
P[6]
P[5]
P[4]
P[3]
P[2]
P[1]
P[0]
IN[15]
IN[14]
IN[13]
IN[12]
IN[11]
IN[10]
IN[9]
IN[8]

Variable Elimination

Identify sub-circuits C_S in the AIG and eliminate internal variables:

- Full-adder rewriting
- Half-adder rewriting
- XOR- Rewriting
- Common-Rewriting

Variable elimination is based on **elimination theory of Gröbner bases**.

Elimination theory of Gröbner bases

Elimination order. Let $X = Y \dot{\cup} Z$ and we want to eliminate Z. Order the terms such that for all terms σ, τ where a variable from Z is contained in σ but not in τ, we obtain $\tau < \sigma$.

Elimination ideal. The elimination ideal J where the Z-variables are eliminated of $I \subseteq \mathbb{Q}[X] = \mathbb{Q}[Y, Z]$ is defined by

$$J = I \cap \mathbb{Q}[Y].$$

Elimination theorem. Given an ideal $I \subseteq \mathbb{Q}[X] = \mathbb{Q}[Y, Z]$. Further let G be a Gröbner basis of I with respect to an elimination order $Y < Z$. Then the set

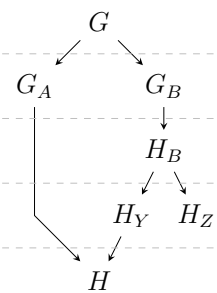
$$H = G \cap \mathbb{Q}[Y]$$

is a Gröbner basis of the elimination ideal $J = I \cap \mathbb{Q}[Y]$, in particular $\langle H \rangle = J$.

Elimination procedure

Problem: Computing a Gröbner basis H for $I(C)$ w.r.t an elimination order is costly.

Solution: Split G into two parts.



Step 1: original Gröbner basis G

Step 2: split G into two subbases

Step 3: change order of $<_G$ to $<_H$

Step 4: eliminate the variables of Z

Step 5: rejoin bases $H = G_A \cup H_Y$

Elimination procedure

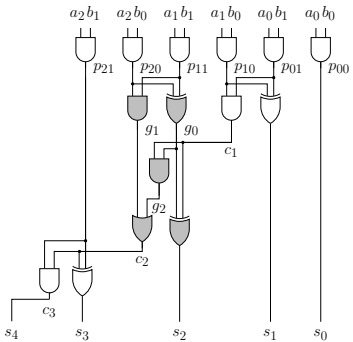
Theorem

Let $G \subseteq \mathbb{Q}[X] = \mathbb{Q}[Y, Z]$ be a Gröbner basis with respect to some term order $<_G$. Let $G_A = G \cap \mathbb{Q}[Y]$ and $G_B = G \setminus G_A$. Let $<_H$ be an elimination order for Z which agrees with $<_G$ for all terms that are free of Z, i.e., terms free of Z are equally ordered in $<_G$ and $<_H$. Suppose that $\langle G_B \rangle$ has a Gröbner basis H_B with respect to $<_H$ which is such that every leading term in H_B is free of Z or free of Y. Then $\langle G \rangle \cap \mathbb{Q}[Y] = (\langle G_A \rangle + \langle G_B \rangle) \cap \mathbb{Q}[Y] = \langle G_A \rangle + (\langle G_B \rangle \cap \mathbb{Q}[Y])$.

Theorem

Let $G, G_A, G_B, H_B, H_Y, H_Z, <_H, <_G$ be as before. Then $H = G_A \cup H_Y$ is a Gröbner basis w.r.t. the ordering $<_H$.

Example: Full-Adder Rewriting



$$G_A = G \backslash G_B$$
$$G_B = \{ -g_0 + p_{20} + p_{11} - 2p_{20}p_{11}, \quad -g_1 + p_{20}p_{11}, \quad -g_2 + c_1 g_0, \\ -s_2 + c_1 + g_0 - 2c_1 g_0, \quad -c_2 + g_1 + g_2 - g_1 g_2 \}$$

Original lexicographic term ordering $<_G$:

$$b_0 < b_1 < a_0 < a_1 < a_2 < p_{00} < s_0 < p_{01} < p_{10} < s_1 < c_1 < \\ p_{11} < p_{20} < g_0 < g_1 < g_2 < s_2 < c_2 < p_{21} < s_3 < c_3 < s_4$$

Elimination order $<_H$:

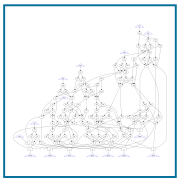
$$b_0 < b_1 < a_0 < a_1 < a_2 < p_{00} < s_0 < p_{01} < p_{10} < s_1 < c_1 < \\ p_{11} < p_{20} < s_2 < c_2 < p_{21} < s_3 < c_3 < s_4 g_0 < g_1 < g_2$$

Gröbner basis H_B w.r.t. elimination order $<_H$:

$$H_B = \{ g_0 + 2p_{20}p_{11} - p_{20} - p_{11}, \quad g_1 - p_{20}p_{11}, \\ g_2 + 2p_{20}p_{11}c_1 - p_{20}c_1 - p_{11}c_1, \\ s_2 - 4p_{20}p_{11}c_1 + 2p_{20}p_{11} + 2p_{20}c_1 - p_{20} + 2p_{11}c_1 - p_{11} - c_1, \\ 2c_2 + s_2 - p_{20} - p_{11} - c_1 \}$$

Experiments

Multiplier



AIG

AIGMULTOPOLY

Polynomials

$$
\begin{aligned}
B = \{ & \\
& x - a_0 * b_0, \\
& y - a_1 * b_1, \\
& s_0 - x * y, \\
& \dots \\
\} &
\end{aligned}
$$

CAS-File

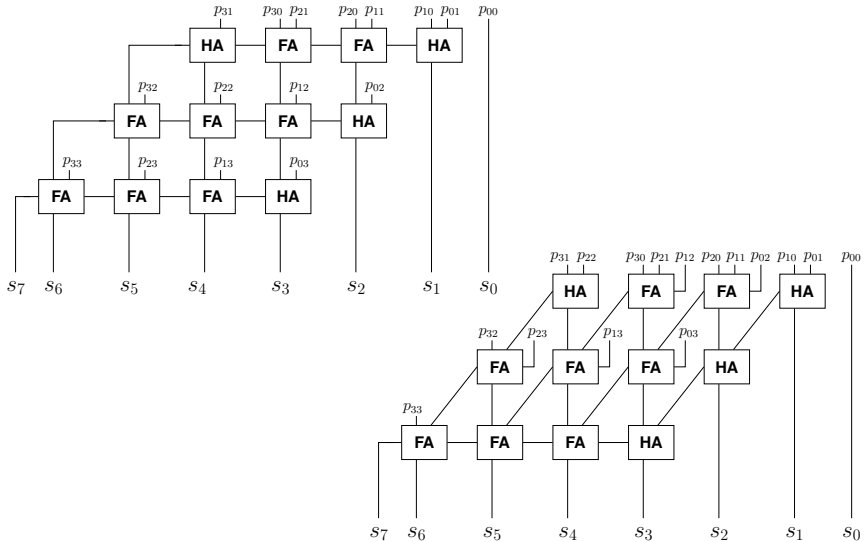
MATHEMATICA
SINGULAR

Ideal Membership

$C_0 = 0$ ✓

$C_0 \neq 0$ ✗

Experiments

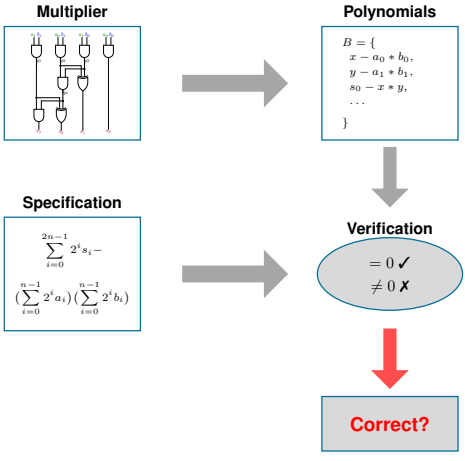


Experiments

mult	n	Mathematica				Singular			
		non-inc	incremental			non-inc	incremental		
				+xor	+xor +add			+xor	+xor +add
btor	16	3	5	2	1	1	1	1	1
btor	32	56	31	14	2	42	28	10	1
btor	64	MO	292	131	11	MO	MO	MO	14
btor	128	TO	TO	TO	101	EE	EE	EE	EE
sp-ar-rc	16	9	7	4	1	TO	6	1	0
sp-ar-rc	32	326	171	30	2	TO	242	28	2
sp-ar-rc	64	MO	TO	300	11	MO	EE	MO	16
sp-ar-rc	128	TO	TO	TO	102	EE	EE	EE	EE

Table: time in sec; TO = 1200 sec, MO = 14GB, EE=more than 32767 variables

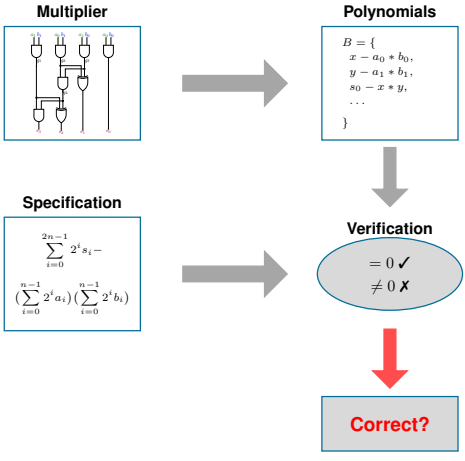
Proofs



Problem:

- Can we trust the CAS?
- Can we trust our own implementation of the optimizations?

Proofs



Problem:

- Can we trust the CAS?
- Can we trust our own implementation of the optimizations?

Solution:

Validate result of verification process [SC2'18]

- Generate machine-checkable proofs
- Check by independent proof checkers

Proofs

Polynomial calculus: Sequence $P = (p_1, \dots p_n)$, where each p_k is obtained by:

Addition	$\dfrac{p_i \qquad p_j}{p_i + p_j}$	$\forall\, p_i, p_j \in \langle G \rangle : p_i + p_j \in \langle G \rangle$
Multiplication	$\dfrac{p_i}{q p_i}$	$\forall\, q \in \mathbb{Q}[X]\ \forall\, p_i \in \langle G \rangle : q p_i \in \langle G \rangle$

Proofs

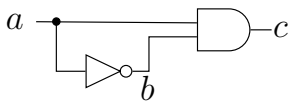
Polynomial calculus: Sequence $P = (p_1, \dots p_n)$, where each p_k is obtained by:

Addition	$\dfrac{p_i \qquad p_j}{p_i + p_j}$	$\forall\, p_i, p_j \in \langle G \rangle : p_i + p_j \in \langle G \rangle$
Multiplication	$\dfrac{p_i}{q p_i}$	$\forall\, q \in \mathbb{Q}[X]\ \forall\, p_i \in \langle G \rangle : q p_i \in \langle G \rangle$

If $p_n = f$ we write $G \vdash f$ or in algebraic terms $f \in \langle G \rangle$.

Refutation: $G \vdash 1$ or $1 \in \langle G \rangle$.

Example



$$
\begin{aligned}
G &= \{ \quad -b+1-a, && b=\neg a \\
&\qquad\; -c+ab, && c=a\wedge b=a\wedge\neg a \\
&\qquad\; a^2-a\} && a\in\mathbb{B} \\
f &= \quad c
\end{aligned}
$$

$$\mathrm{red}_G(f) = 0$$

$$
\begin{array}{lcl}
 & * & \dfrac{-b+1-a}{-ab+a-a^2} \\
+ & & \dfrac{a^2-a}{\quad + \dfrac{-ab}{\quad * \dfrac{-c}{c}} \qquad -c+ab}
\end{array}
$$

$$P = (-ab+a-a^2,\ -ab,\ -c,\ c)$$

Practical Algebraic Calculus

We translate the polynomial calculus into a more concrete proof format:

- For correctness it is important to know how the polynomials in the proof where derived
- Usually known $\rightarrow$ store this information

Practical Algebraic Calculus (PAC)

allows automated proof checking

PAC Syntax

letter ::= ‘a’ | ‘b’ | ... | ‘z’ | ‘A’ | ‘B’ | ... | ‘Z’

number ::= ‘0’ | ‘1’ | ... | ‘9’

constant ::= (number)$^+$

variable ::= letter (letter | number)*

power ::= variable [‘^’ constant]

term ::= power (‘*’ power)*

monomial ::= constant | [constant ‘*’] term

operator ::= ‘+’ | ‘-’

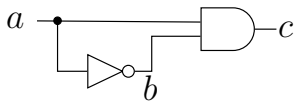
polynomial ::= [‘-’] monomial (operator monomial)*

given ::= (polynomial ‘;’)*

rule ::= (‘+’ | ‘*’) ‘:’ polynomial ‘,’ polynomial ‘,’ polynomial ‘;’

proof ::= (rule ‘;’)*

Example - PAC



$$G = \{ \; -b+1-a, \; -c+ab, \; a^2-a \}$$

$$f = c$$

```
* : -b+1-a,        a,          -a*b+a-a^2;
+ : -a*b+a-a^2,    a^2-a,      -a*b;
+ : -a*b,          -c+a*b,     -c;
* : -c,            -1,         c;
```

Proof Checking

A proof **rule** contains four components:

$$o : v, w, \ p;$$

Proof checking:

- Connection property: v, w are given polynomials or conclusions p_i of previous rules
- Inference property: verify correctness of each rule, e.g. $p = v + w$ for $o = " + "$
- Refutation check: at least one p_i is a non-zero constant

Proof Checking Algorithm

input G sequence of given polynomials
$r_1 \cdots r_k$ sequence of PAC proof rules

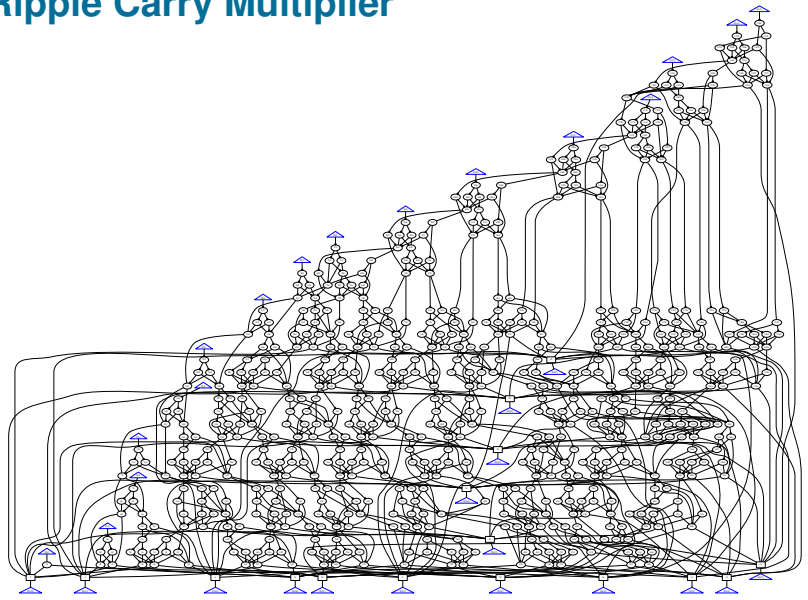
output "incorrect", "correct-proof", or "correct-refutation"

$P_0 \leftarrow G$
for $i \leftarrow 1 \ldots k$
  let $r_i = (o_i, v_i, w_i, p_i)$
  case $o_i = +$
    if $v_i \in P_{i-1} \ \wedge \ w_i \in P_{i-1} \ \wedge \ p_i = v_i + w_i$ **then** $P_i \leftarrow \text{append}(P_{i-1}, p_i)$
    else return "incorrect"
  case $o_i = *$
    if $v_i \in P_{i-1} \ \wedge \ p_i = v_i * w_i$ **then** $P_i \leftarrow \text{append}(P_{i-1}, p_i)$
    else return "incorrect"
for $i \leftarrow 1 \ldots k$
  if p_i is a non zero constant polynomial **then return** "correct-refutation"
return "correct-proof"

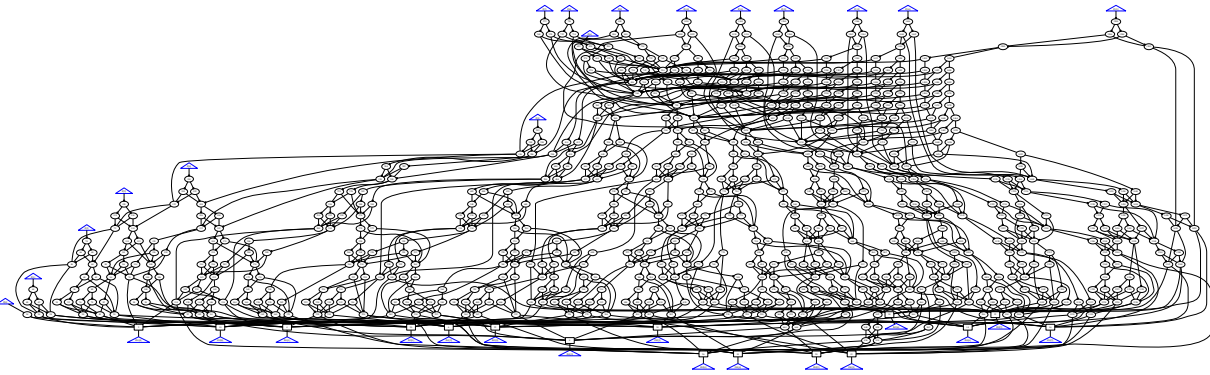
Verifying Correctness of FMCAD'17 and DATE'18 Results

n	mult	option	verification	proof generation	PACTRIM	length
16	btor	incrm+elim	1	37	0	12738
32	btor	incrm+elim	2	801	1	53122
64	btor	incrm+elim	11	22378	4	216834
16	sparrc	incrm+elim	1	112	20	17804
32	**sparrc**	**incrm+elim**	**2**	**2611**	**2**	**75244**
64	sparrc	incrm+elim	11	80906	12	309164

Array Ripple Carry Multiplier

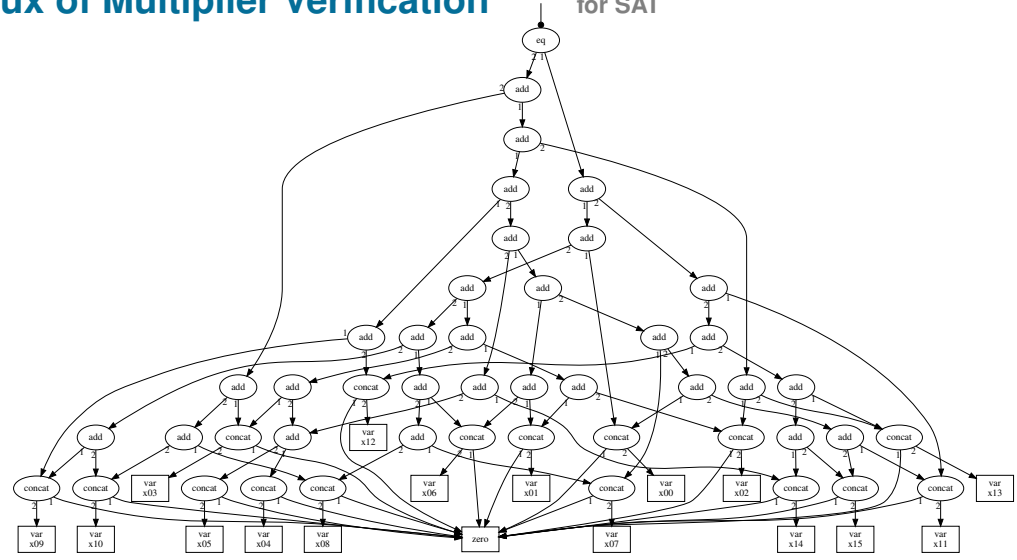


Wallace-Tree Carry-Lookahead Multiplier



Crux of Multiplier Verification

for SAT



Verification and Certification Time FMCAD'19 Submission

architecture	n	SPEC	nosfx	nomod	noelim	Verify				**MG**D	YC**M**	CSYY	R**B**K	Certify				Check			total	proof size	
						sub	sfx	aig	tot	DAC19	TCAD17	TCAD19	FMCAD17	sub	sfx	aig	tot	sfx	aig	tot		sfx	aig
sp-ar-rc	32	u	0	0	0	0	0	0	0	NA_2	NA_4	0	8	0	0	1	1	0	1	1	2	0	46 018
sp-dt-lf	32	u	TO	0	1	0	0	0	0	3	TO	NA_4	TO	0	0	1	1	0	1	1	2	2 614	44 986
sp-wt-cl	32	u	TO	TO	1	0	1	0	1	6	TO	NA_4	TO	0	1	1	2	1	1	2	4	52 390	47 017
sp-bd-ks	32	u	TO	TO	TO	0	0	0	1	12	TO	NA_4	TO	0	0	1	1	0	1	1	2	21 188	46 971
sp-ar-ck	32	u	TO	0	0	0	0	0	0	5	TO	NA_4	TO	0	0	1	1	0	1	1	2	663	45 514
bp-ar-rc	32	u	0	TO	264	0	0	0	0	4	0	NA_4	TO	0	0	1	1	0	1	1	2	0	41 935
bp-ct-bk	32	u	TO	TO	210	0	0	0	0	9	TO	NA_4	TO	0	0	1	1	0	1	1	2	1 832	35 803
bp-os-cu	32	u	0	TO	272	0	0	0	0	7	TO	NA_4	TO	0	0	1	1	0	1	1	2	0	44 528
bp-wt-cs	32	u	0	TO	257	0	0	0	0	5	TO	NA_4	TO	0	0	1	1	0	1	1	2	0	42 963
sp-ar-rc	64	u	2	1	5	0	0	2	2	NA_2	NA_4	0	133	0	0	13	13	0	6	6	19	0	188 290
sp-dt-lf	64	u	TO	2	13	0	0	3	3	31	TO	NA_4	TO	0	0	17	17	0	8	8	25	34 423	186 170
sp-wt-cl	64	u	TO	TO	9	0	9	2	11	96	TO	NA_4	TO	0	9	14	23	7	6	13	37	264 471	191 621
sp-bd-ks	64	u	TO	TO	7	0	1	2	3	162	TO	NA_4	TO	0	2	14	16	1	6	7	23	78 567	190 911
sp-ar-ck	64	u	TO	1	5	0	0	2	2	143	TO	NA_4	TO	0	0	13	13	0	6	6	19	1 432	187 251
bp-ar-rc	64	u	2	TO	TO	0	0	2	2	53	0	NA_4	TO	0	0	15	15	0	5	5	20	0	161 935
bp-ct-bk	64	u	TO	TO	TO	0	0	2	3	119	TO	NA_4	TO	0	0	16	17	0	5	5	22	27 552	138 179
bp-os-cu	64	u	3	TO	TO	0	0	3	4	95	TO	NA_4	TO	0	0	18	18	0	6	7	25	0	166 963
bp-wt-cs	64	u	3	TO	TO	0	0	3	3	75	TO	NA_4	TO	0	0	15	15	0	5	5	20	0	161 745
sp-ar-rc	32	s	0	0	0	0	0	0	0	NA_1	NA_1	NA_1	NA_1	0	0	1	1	0	1	1	2	0	46 090
bp-wt-cl	32	s	TO	0	245	0	1	0	1	NA_1	NA_1	NA_1	NA_1	0	1	1	2	1	1	2	4	50 620	38 097
btor	32	t	0	NA_3	0	0	0	0	0	NA_1	NA_1	NA_1	NA_1	0	0	0	0	0	0	1	1	0	16 633
sp-ar-rc	64	s	2	1	5	0	0	2	2	NA_1	NA_1	NA_1	NA_1	0	0	13	13	0	6	6	19	0	188 426
bp-wt-cl	64	s	TO	47	TO	0	10	3	12	NA_1	NA_1	NA_1	NA_1	0	10	17	26	7	5	12	38	261 650	151 355
btor	64	t	1	NA_3	1	0	0	1	1	NA_1	NA_1	NA_1	NA_1	0	0	4	4	0	3	4	8	0	69 068

NA_1: tool not applicable to type SPEC NA_2: tool not yet available NA_3: would lead to incompleteness NA_4: not reproducible

Large Multipliers FMCAD'19 Submission

architecture	n	SPEC	Verify				**MG**D	CSYY	AIG size
			sub	sfx	aig	tot	DAC19	TCAD19	
btor	128	u	0	0	34	34	NA_2	0	129 920
kjvnkv	128	u	0	0	19	19	NA_2	NA_4	194 560
sp-ar-rc	128	u	0	0	19	19	349	2	194 560
sp-dt-lf	128	u	0	2	48	50	490	NA_4	193 550
sp-wt-bk	128	u	0	1	54	56	746	NA_4	197 518
btor	256	u	1	0	536	537	NA_2	NA_4	521 984
kjvnkv	256	u	1	0	252	253	NA_2	NA_4	782 336
sp-ar-rc	256	u	2	0	254	255	8 720	NA_4	782 336
sp-dt-lf	256	u	4	6	901	911	12 874	NA_4	780 302
sp-wt-bk	256	u	3	3	985	992	21 454	NA_4	790 098
btor	512	u	7	0	10 458	10 465	NA_2	NA_4	2 092 544
kjvnkv	512	u	9	0	4 592	4 601	NA_2	NA_4	3 137 536
sp-ar-rc	512	u	10	0	4 541	4 551	192 640	NA_4	3 137 536
sp-dt-lf	512	u	26	21	21 563	21 609	240 051	NA_4	3 133 454
sp-wt-bk	512	u	25	9	48 197	48 231	492 320	NA_4	3 156 866
btor	1024	u	96	0	306 157	306 253	NA_2	NA_4	8 379 392
kjvnkv	1024	u	107	0	97 670	97 777	NA_2	NA_4	12 566 528

NA_1: tool not applicable to type SPEC NA_2: tool not yet available
NA_3: would lead to incompleteness NA_4: not reproducible

Future Work

Circuit Verification

- other word-level operators (shift, division, . . .)
- more complex multipliers
- truncated multipliers
- negative numbers

Proof Generation

- connection to clausal proof systems
- certified proof checker — really really correct
- boolean proofs

ALGEBRA, PROOFS AND MULTIPLIERS

Armin Biere joint work with Daniela Kaufmann and Manuel Kauers

28th International Workshop on Logic & Synthesis

EPFL – Lausanne, Switzerland, June 22, 2019

FWF Der Wissenschaftsfonds.

RiSE Rigorous Systems Engineering

JKU JOHANNES KEPLER UNIVERSITÄT LINZ